Translator's Introduction

Euler wrote *Recherches sur les racines imaginaires des équations* while at the Berlin Academy, and it is found in the *Mémoires de l'académie des sciences de Berlin*, 1751, pages 222–288.

In the first part of this article, Euler concerns himself with what today we call the Fundamental Theorem of Algebra, or as Euler says in section 49,

Every rational function of a variable x, as

$$x^m + Ax^{m-1} + Bx^{m-2} + \cdots$$

can always be resolved into real factors, either simple of the form $x + p$, or else double of the form $xx + px + q$.

Intimately related to this is the idea of imaginary numbers, which Euler treats in depth.

Euler works out the factorization for $x^4 + 2x^3 + 4x^2 + 2x + 1$ using clever, though accessible, algebra. Then he works out the factorization for a more general degree 4 equation. He discusses equations of odd and even degree, and shows how the number of real and imaginary factors relates to the parity of the degree of the equation. He continues by considering a large number of special cases, discussing each one in detail and relating them to each other.

Others have found that Euler did not completely sew up the matter in his proof, and indeed a complete proof of the Fundamental Theorem of Algebra that satisfies modern standards did not occur until over a century after this article was written.

Nevertheless, the reader will be well rewarded for following along as Euler works through this problem. There is much skillful algebra, and it is interesting to see basic results intermixed with more advanced manipulations. Euler is simply telling you what he is thinking.

Euler says the proof is complete in section 49, and we can perhaps detect a slight degree of unease when he writes that "in case one wanted to have trouble recognizing the correctness of these proofs, I am going to add several propositions concerning this subject that will not depend on the preceding, and whose truth will serve to lift any doubt that one might still have." Euler then offers additional proofs of some of the special cases.

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This flows into the second half of the article, where Euler discusses the ways a quantity can be imaginary. For Euler, imaginary means "neither greater than zero, nor less than zero, nor equal to zero," and he gives as an example $\sqrt{-1}$, or more generally $a + b\sqrt{-1}$. So Euler addresses the question of whether there might be other ways a quantity can be imaginary, perhaps without being reducible to this form.

To this end, Euler shows that when you apply the common operations of analysis—addition, subtraction, multiplication, and division—to quantities of the canonical form, the result can be reduced to the canonical form.

He then goes further and considers whether perhaps transcendental operations might yield quantities that are imaginary in some distinct way, and shows that the known transcendental operations—those involving logarithms, angles, and the like—all yield quantities that can be reduced to the canonical form.

This involves a remarkably clear and insightful explanation of raising real and imaginary quantities to real and imaginary powers, of the trigonometric functions applied to imaginary arguments, and of logarithms and antilogarithms of imaginary quantities.

Euler's purpose is to demonstrate that all of these operations yield quantities, often an infinity of quantities, that can be written in the canonical form. But the results themselves and Euler's explanation of them offer much to the contemporary reader who is interested in what might be meant by these kinds of expressions. As just one example, Euler observes that the expression $\sqrt{-1}$ raised to $\sqrt{-1}$ has an infinity of values, and that, surprisingly, they are all real. More generally, the periodic aspect of the transcendental operations when applied to imaginary quantities is explained quite clearly. Many of the familiar identities involving e and the trigonometric functions are laid out.

The ideas in this article are for the most part algebraic, but not exclusively so. Intuitive use of the intermediate value theorem enters into his proof early on, when he argues that every odd-degree equation has at least one real root, a result which he uses repeatedly. And in the second part of the article, when Euler shows how to find all the roots of an imaginary quantity $M + Ni$ his exposition is tantalizingly close to what we think of as the complex plane, an idea that was articulated in modern form about a half century later.

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The reader who is new to Euler might be surprised to discover how similar Euler's notation is to our own, and how modern his view of mathematics is.

Euler does use the terms "simple factor" and "double factor" where we might say "linear factor" and "quadratic factor", but after one reads it Euler's way, one might wonder why we do it differently.

Euler tends to write xx instead of x^2, and one gets used to that quickly enough. This was probably done to make the typesetting easier, and was common at the time.

In this translation, I have sometimes written i where Euler has written $\sqrt{-1}$. We must note, however, that this substitution is not completely unproblematical, and should not be done mechanically. Euler knew very well, and articulates in several places in this article, that $\sqrt{-1}$ has a dual nature, that every quantity has two square roots, -1 included.

So when Euler is referring to that dual nature, it can be odd, or even misleading, to replace $\sqrt{-1}$ with i. At other times, Euler's use of $\sqrt{-1}$ is close enough, although not identical to, our notion of i that it seems justified to use i, in order not to distract the contemporary reader. But the reader should note that, in this article, Euler never wrote i, but always wrote $\sqrt{-1}$.

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Euler writes in a clear and direct way, as if speaking to the reader. There is a complete lack of pretense and affectation. Euler wrote for the interested, informed reader, but he did not assume that the reader already knew what he had to say.

To express his ideas, Euler tends to use active verbs in preference to nouns, and he tends to focus on the operations and activities involved, and not on the definitions. He defines things as he needs them, and his definitions tend to be simple.

Euler manipulates linguistic expressions with the same ease that he manipulates mathematical expressions, and while his sentences are sometimes long and complex, they never ramble. Throughout, the reader will perceive a sense of enthusiasm and discovery, and because Euler is as generous as he is skilled, and because of the tone and manner of the writing, the reader is led to feel as though he is in on the discovery himself.

None of this occurs with the slightest degree of condescension or oversimplification, and Euler does credit the reader with an attention span equal to his own (as, for example, in the detailed treatment of the many special cases in the first part of the article). Still, interesting and beautiful results sometimes pop out as if from nowhere, and the reader wonders how that happened. But it is all right there. The reader has experienced an "Euler moment," and wishes that more mathematical writing were like this.

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