A New Method for Eliminating Unknown Quantities from Equations

Leonhard Euler

1. When, in order to solve a problem, we are obliged to introduce into the calculation several unknown quantities, the solution likewise leads to several equations, from which we must then find the values of each of these unknown quantities. This is done by means of elimination. We start with one of the unknown quantities, regarding the others as known. Then we try to find its value by using one or more of the given equations to derive it, expressing it by a rational formula, and as simply as we can. Then we substitute this value into the other equations, and by this means both the number of the unknowns and the number of equations will become smaller by one. In the same manner, we then eliminate another unknown, and we continue these operations until there remains in the calculation only a single equation, the resolution of which furnishes the solution to the problem.

2. Now, having several equations, each of which contains the unknown quantity we want to eliminate, we see first that we might be able to take just one of them in order to find the value of this unknown (which when substituted into the other equations would render the both the number of unknowns and equations smaller by one). This route is indeed very appropriate when the unknown to eliminate does not hold more than one dimension in the equation that we chose in order to derive from it its value. But, if the unknown rises to two or more dimensions, we would not often be in a position to find its value, and even if we were, the irrational value which we might obtain could lead to extremely troublesome calculations, often rendering the solution impracticable.

3. So, when no single equation is found where the unknown that we want to eliminate has only a single dimension, we must chose two of them in order to derive the value. For it has been demonstrated that however many dimensions the unknown may have in two equations, it is always possible to successively decrease the dimensions, and this until we reach one equation which doesn't contain this unknown at all. In the same way, by combining

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two other equations, we will derive a new one which will no longer contain this unknown. And so we will form as many equations free of this unknown as necessary, in order to eliminate by a similar method the other unknowns, until we reach a single equation which furnishes the solution to the given problem.

4. The method of elimination reduces, then, to the case of two equations which both contain the quantity we intend to eliminate; and all the work comes down to finding one equation which no longer contains this quantity. We clearly see that the operation to reach this goal will become all the more difficult as the dimensions of the quantity we want to eliminate in the two equations rise, at least when no very specific circumstance lessens the work. And so that we are not obliged to do this operation for each proposed case, we find in the *Arithmetica universalis* of Mr. Newton some formulas appropriate to this purpose, with the help of which the elimination can be easily done, even when the quantity to eliminate rises, in the two equations, up to four dimensions. For Mr. Newton takes two general equations which do not surpass this degree, and he returns the equation which results after the elimination, so that we need only apply it to each proposed case. Before explaining my new method, it would be appropriate to give an idea of the one Mr. Newton appears to have used.

5. I will start with two equations, where the quantity to eliminate, z, rises only to one dimension, these being

$$A + Bz = 0,$$

and

$$a + bz = 0,$$

so that we see more clearly how the operations multiply when going to higher equations. So first of all, it is clear that we have only to multiply the first equation by b and the other by B, because subtracting the latter product from the former, we will have

$$Ab - Ba = 0,$$

which is the equation that results by elimination of the quantity z. We could also multiply the first by a and the other by A, so that after the subtraction of one from the other the constant terms cancel, and then we will have

$$Baz - Abz = 0,$$

which after dividing by z, gives as before

$$Ba - Ab = 0 \quad \text{or} \quad Ab - Ba = 0.$$

6. Now let the two following equations be proposed, where the quantity to be eliminated, z, rises to two dimensions,

$$A + Bz + Czz = 0,$$

and

$$a + bz + czz = 0.$$

We multiply the first by c and the other by C, and the difference will be

$$Ac - Ca + (Bc - Cb)z = 0.$$

Then, we multiply the first by a and the other by A, and the difference after being divided by z will be

$$Ba - Ab + (Ca - Ac)z = 0.$$

We have two equations where the quantity z rises only to one dimension, so this case now reduces to the preceding, and therefore the elimination will be made by the formula found above, and will give

$$(Ac - Ca)(Ca - Ac) - (Bc - Cb)(Ba - Ab) = 0,$$

or by changing the signs

$$AAcc - 2ACac + CCaa + BBac - ABbc - BCab + ACbb = 0.$$

7. If the two proposed equations are cubics:

$$A + Bz + Czz + Dz^3 = 0,$$

and

$$a + bz + czz + dz^3 = 0,$$

multiplying the first by d and the other by D, the difference will be

$$Ad - Da + (Bd - Db)z + (Cd - Dc)zz = 0.$$

Now multiplying the first by a and the other by A, the difference after dividing by z will give

$$Ba - Ab + (Ca - Ac)z + (Da - Ad)zz = 0.$$

We have therefore reached two square equations, for which we eliminate the quantity z by the preceding section. In the same way, if the two proposed equations are of the fourth degree, we reduce them to two cubic equations, and in general whatever the degree of the two first equations, we reduce them to two equations of one degree less. Continuing this reduction, then, we will reach the end necessarily with one equation, which will no longer contain the quantity z.

8. To make this elimination easier for the two cubic equations

$$A + Bz + Czz + Dz^3 = 0,$$

and

$$a + bz + czz + dz^3 = 0,$$

we will make the following substitutions

$$\begin{aligned} Ad - Da &= A' \qquad & aB - bA &= a' \\ Bd - Db &= B' \qquad & aC - cA &= b' \\ Cd - Dc &= C' \qquad & aD - dA &= c' \end{aligned}$$

and the square equations will be

$$A' + B'z + C'zz = 0$$

and

$$a' + b'z + c'zz = 0.$$

Then, we additionally put

$$\begin{aligned} A'c' - C'a' &= A'' \qquad & a'B' - b'A' &= a'' \\ B'c' - C'b' &= B'' \qquad & a'C' - c'A' &= b'' \end{aligned}$$

in order to get these two simple equations

$$A'' + B''z = 0$$

and

$$a'' + b''z = 0,$$

and the desired equation, which no longer contains z, will be

$$A''b'' - B''a'' = 0.$$

9. If we count the number of letters A, B, C, D, a, b, c, d, which are found multiplied together in each term, we see that the expressions marked A', B', C', a', b', c', contain two dimensions, and therefore the letters A'', B'', a'', b'' contain four, so that the last equation $A''b'' - B''a'' = 0$ will be of 8 dimensions, or each term will be composed of 8 letters. Now, in developing this equation, we find that it is divisible by $Ad - Da$, so that it is only of 6 dimensions, namely

$$\begin{aligned}(Ad - Da)^3 + (Ac - Ca)^2(Cd - Dc) - 2(Ab - Ba)(Ad - Da)(Cd - Dc)&\\ + (Bd - Db)^2(Ab - Ba) \quad - (Ab - Ba)(Bc - Cb)(Cd - Dc)&\\ - (Ad - Da)(Ac - Ca)(Bd - Db)&\\ = 0.&\end{aligned}$$

If the two proposed equations are of the fourth degree, this method will lead to an equation of 16 dimensions, but which will be reduced to 8 dimensions since it will be divisible by a formula of 8 dimensions, and so on.

10. We see then that this method often leads to some overly complicated equations, which contain factors altogether useless for the purpose we have in view. For in the case of cubic equations, it is evident that the factor $Ad - Da$ does not satisfy the question, since the elimination might not lead to this equation $Ad - Da = 0$. Therefore, as this factor is contained in the final equation, we cannot regard it as exactly right; since an equation of several dimensions does not furnish an exact solution to a problem unless all the roots satisfy the conditions of the problem. For not knowing how to discern the false roots from the true ones, we risk falling into a completely false solution. So although the equations we get by following this method contain the true solution, they also often contain false solutions, and this is a very considerable shortcoming.

11. This situation gave me occasion to look for another method of elimination, which would be free of this fault and would at the same time be based on the nature of equations in such a way that we might understand more clearly the reason for all the operations we are obliged to do. Now first, since the idea of elimination seems not sufficiently precise, I will begin by better developing this idea and by determining more exactly what this question reduces to. For as soon as we form a correct idea about the purpose for which we employ elimination, we will see, first, which operations we will be obliged to undertake in order to achieve this end. Furthermore, we will be in a position to give this investigation a larger scope and to apply it to several other questions which could be useful in Analysis and in the Theory of Curves.

12. In order to make the reasoning more intelligible, I will first consider only one particular case, where the quantity to eliminate, z, rises in one equation to the third degree and in the other to the second. So let these two equations be

$$zz + Pz + Q = 0$$

and

$$z^3 + pzz + qz + r = 0,$$

where the letters P, Q, p, q, r contain the other unknown quantities. We want to know the relation which will remain among these other unknowns after we have eliminated the quantity z. This relation will be contained in one equation, which we reach by eliminating z; and this equation will contain the letters P, Q, p, q, r and will consequently determine their mutual relationship, in order that the two given equations may hold. But, for these two equations to simultaneously hold, there must be a certain value, which when put for z, causes both of these formulas

$$zz + Pz + Q$$

and

$$z^3 + pzz + qz + r,$$

to vanish. That is to say, it is necessary that the two given equations have a common root, which belongs equally to one and the other.

13. Notice then what elimination of the quantity z reduces to: it is to determine a relation among the coefficients, or the quantities P, Q, p, q, r, such that the two given equations obtain a common root. Let ω be the value of this common root, and $z - \omega$ will be a factor of both of them, so that we can put

$$zz + Pz + Q = (z - \omega)(z + \mathfrak{A}),$$

$$z^3 + pzz + qz + r = (z - \omega)(zz + \mathfrak{a}z + \mathfrak{b}),$$

and from there it is clear that we must have

$$(zz + Pz + Q)(zz + \mathfrak{a}z + \mathfrak{b}) = (z^3 + pzz + qz + r)(z + \mathfrak{A}).$$

Now, by equating the two products, we will get four equalities:

$$\begin{array}{rlrl} \text{I.} & P + a = p + \mathfrak{A} & \text{II.} & Q + P\mathfrak{a} + \mathfrak{b} = q + p\mathfrak{A} \\ \text{III.} & P\mathfrak{b} + Q\mathfrak{a} = q\mathfrak{A} + r & \text{IV.} & Q\mathfrak{b} = r\mathfrak{A} \end{array}$$

from which we will easily determine the three new letters $\mathfrak{A}$, $\mathfrak{a}$, and $\mathfrak{b}$, and then obtain the equation sought, which contains the required relation among the coefficients P, Q, p, q, r, or will be what we find by elimination of z.

14. This determination will not be any obstacle, since we only have to resolve simple equations. For the first equality gives

$$\mathfrak{A} = P - p + \mathfrak{a},$$

and the second

$$\mathfrak{b} = q + p\mathfrak{A} - Q - P\mathfrak{a},$$

or

$$b = q + Pp - pp + p\mathfrak{a} - Q - P\mathfrak{a},$$

and these values when substituted in the third equality give

$$Pq + PPp - Ppp - PQ + P(p - P)\mathfrak{a} + Q\mathfrak{a} = Pq - pq + q\mathfrak{a} + r,$$

or

$$Pp(P - p) + pq - PQ - r = P(P - p)\mathfrak{a} - (Q - q)\mathfrak{a},$$

from which we derive

$$\mathfrak{a} = \frac{Pp(P - p) + pq - PQ - r}{P(P - p) - (Q - q)} = p - \frac{Q(P - p) + r}{P(P - p) - (Q - q)}.$$

Now the same values give for the fourth equality

$$Qq + PQp - Qpp - QQ + Q(p - P)\mathfrak{a} = r(P - p) + r\mathfrak{a},$$

or

$$\mathfrak{a} = \frac{Qp(P - p) - Q(Q - q) - r(P - p)}{Q(P - p) + r} = p - \frac{Q(Q - q) + Pr}{Q(P - p) + r}.$$

Therefore, equating these two values, we get

$$\frac{Q(P-p)+r}{P(P-p)-(Q-q)} = \frac{Q(Q-q)+Pr}{Q(P-p)+r},$$

or

$$\begin{gathered} Q(P-p)(Pq-Qp)+2Qr(P-p)+Pr(Q-q)-PPr(P-p) \\ +Q(Q-q)^2+rr=0. \end{gathered}$$

15. Now it is clear how we must proceed in order to eliminate the unknown z from two given equations of arbitrary degree. For, let the two general equations be:

$$\begin{aligned} z^m + Pz^{m-1} + Qz^{m-2} + Rz^{m-3} + Sz^{m-4} + \cdots &= 0, \\ z^n + \ pz^{n-1} + \ qz^{n-2} + \ rz^{n-3} + \ sz^{n-4} + \cdots &= 0, \end{aligned}$$

where we must furnish an equation which no longer contains the quantity z. This question comes down to determining the relation among the coefficients P, Q, R, etc., p, q, r, etc., so that the two given equations obtain a common root, or indeed a common factor. Let $z-\omega$ be this common factor, and we will put

$$\begin{aligned} z^m + Pz^{m-1} + Qz^{m-2} + \cdots &= (z-\omega)(z^{m-1} + \mathfrak{A}z^{m-2} + \mathfrak{B}z^{m-3} + \cdots), \\ z^n + \ pz^{n-1} + \ qz^{n-2} + \cdots &= (z-\omega)(z^{n-1} + \ \mathfrak{a}z^{n-2} + \ \mathfrak{b}z^{n-3} + \cdots). \end{aligned}$$

16. We will then have to equate the two following products:

$$(z^m + Pz^{m-1} + Qz^{m-2} + \cdots)(z^{n-1} + \ \mathfrak{a}z^{n-2} + \ \mathfrak{b}z^{n-3} + \cdots)$$

and

$$(z^n + \ pz^{n-1} + \ qz^{n-2} + \cdots)(z^{m-1} + \mathfrak{A}z^{m-2} + \mathfrak{B}z^{m-3} + \cdots),$$

and since the first terms are already equal, the number of equalities that we will derive will be equal to $m+n-1$. Now the number of letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, etc., is $m-1$, and the number of letters $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, etc., is $n-1$, so the number of all these letters together, whose values we must find, will be $m+n-2$; and for this purpose as many equations would suffice. Since we have one equation in excess, we will at the end reach one equation, which will not contain any of the letters $\mathfrak{A}$, $\mathfrak{B}$, etc., $\mathfrak{a}$, $\mathfrak{b}$, etc., and since z will not be found there either, this will be the equation sought, to which the elimination leads; or which contains the required relation among the coefficients P, Q, R, etc., p, q, r, etc., so that the two given equations obtain a common root.

17. Having then put into full light the nature of elimination, and of the operations we must execute for this purpose, it will be easy to use them in any given case. To give one example, I will relate a problem proposed in the *Actes de Leipzig*, in the month of October 1749, which gives a fourth degree equation

$$x^4 = pxx + qx + r,$$

where the second term is missing, and we are to find another

$$x^4 = fx^3 + gxx + hx + r,$$

which contains the second term, and where the last term is the same as the given one, and which shares with the first a common root. Or, we must find the equation which results from eliminating from these two equations the quantity x; for this equation will contain the relation that the coefficients f, g, h must have with respect to the given quantities p, q, r, so that these two equations will obtain a common root.

18. To solve this problem, we have only to resolve this equation

$$\begin{aligned}(x^4 - pxx - qx - r)&(x^3 + Axx + Bx + C)\\ &= (x^4 - fx^3 - gx^2 - hx - r)(x^3 + Dxx + Ex + F),\end{aligned}$$

from which we derive the following equalities:

$$\begin{aligned} A &= D - f,\\ B - p &= E - Df - g,\\ C - Ap - q &= F - Ef - Dg - h,\\ -Bp - Aq - r &= -Ff - Eg - Dh - r,\\ -Cp - Bq - Ar &= -Fg - Eh - Dr,\\ -Cq - Br &= -Fh - Er,\\ -Cr &= -Fr.\end{aligned}$$

The first two with the last will give, to start

$$\begin{aligned} &A = D - f,\\ &B = E - Df - g + p,\\ &C = F,\end{aligned}$$

which values, when substituted into the others, will produce

$$\begin{aligned}
&Dp - fp + q - Ef - Dg - h = 0,\\
&Ep - Dfp - gp + pp + Dq - fq - Ff - Eg - Dh = 0,\\
&Fp + Eq - Dfq - gq + pq - fr - Fg - Eh = 0,\\
&Fq - Dfr - gr + pr - Fh = 0.
\end{aligned}$$

19. The first and the last of these inequalities furnish

$$E = \frac{D(p-g)}{f} - p + \frac{q-h}{f} \qquad \text{and} \qquad F = \frac{Dfr - r(p-g)}{q-h}$$

and from there the two other equalities will take the following forms:

$$\begin{aligned}
&Df^3r + Df^2p(q-h) - Df(q-h)^2 - D(p-g)^2(q-h)\\
&\qquad = (p-g)(q-h)^2 - f^2q(q-h) + f^2r(p-g),\\
&D(p-g)(q-h)^2 - Df^2q(q-h) + Df^2r(p-g)\\
&\qquad = f^2r(q-h) + fr(p-g)^2 - fhp(q-h) + fgq(q-h) - (q-h)^3,
\end{aligned}$$

from which we finally derive this equation:

$$\begin{aligned}
f^4rr - f^3r(gq + 2hp - 3pq) - 2f^2r(q-h)^2 \quad &- 4fr(p-g)^2(q-h)\\
- f^3qq(q-h) \qquad + \ f^2pr(p-g)^2 \ &+ \ f(pq + 2hp - 3gq)(q-h)^2\\
- \ f^2p(hp - gq)(q-h)&\\
= r(p-g)^4 - (hp - gq)(p-g)^2(q-h) - (q-h)^4.&
\end{aligned}$$

20. We have then a fourth-degree equation to resolve if we would determine f, or g, or h, in order that the equation

$$x^4 = fx^3 + gxx + hx + r$$

have a common root with the given equation

$$x^4 = pxx + qx + r.$$

But, if we wanted to determine the constant term, r, common to these two equations, regarding the other coefficients f, g, h, p, q as knowns, this could be done by resolution of a square equation. We could even suppose in advance that $f=0$ and determine either of the coefficients g or h, so that these two equations

$$x^4 = gxx + hx + r \quad \text{and} \quad x^4 = pxx + qx + r$$

obtain a common root, which will happen by satisfying this equation:

$$r(p-g)^4 = (hp-gq)(p-g)^2(q-h) + (q-h)^4,$$

from which we see that can be done unless $g=p$ or $h=q$.

21. But the method which I just explained extends much further than simply the work of elimination, and we can by its aid resolve a great many problems which could be very important, both in Analysis and in the Theory of Curves. It is also in this connection that I believe this method merits some attention. Because, if it were limited only to the operations of elimination, I acknowledge that the preference which it would merit over the other methods found for this purpose would not be very considerable, if it were only that it better reveal to us the nature of elimination. Here, then, is another problem, for the resolution of which this method can be employed.

Given two indeterminate algebraic equations, find the determinations necessary for these equations to obtain two common roots.

22. Let one of these two equations be of the third degree and the other of the fourth degree:

$$z^3 + Pzz + Qz + R = 0$$

and

$$z^4 + pz^3 + qzz + rz + s = 0,$$

where we ask what relation must exist between the coefficients so that these two equations will have two roots, or two simple factors, in common. Let $z+\alpha$ and $z+\beta$ be these two common factors, and the two equations must have the following forms:

$$z^3 + Pzz + Qz + R = (z+\alpha)(z+\beta)(z+A)$$
$$z^4 + pz^3 + qzz + rz + s = (z+\alpha)(z+\beta)(zz+az+b),$$

from which we will first derive this:

$$(z^3 + Pzz + Qz + R)(zz+az+b) = (z^4 + pz^3 + qzz + rz + s)(z+A),$$

where each power of z must be set equal.

23. From there we will derive the following five equalities:

$$\begin{aligned}P + a &= p + A,\\ Q + aP + b &= q + Ap,\\ R + aQ + bP &= r + Aq,\\ aR + bQ &= s + Ar,\\ bR &= As.\end{aligned}$$

The first and the last give

$$a = p + A - P \quad \text{and} \quad b = \frac{As}{R},$$

and these values when substituted into the three others:

$$\begin{aligned}A(PR - pR + s) &= PR(P - p) - R(Q - q),\\ A(QR - qR + Ps) &= QR(P - p) - R(R - r),\\ A(RR - Rr + Qs) &= RR(P - p) + Rs.\end{aligned}$$

From there we derive, by eliminating A, these two equations:

$$\begin{aligned}0 = ss + 2Rs(P - p) - PQs(P - p) + Qs(Q - q) + R(P - p)(Pr - rP)&\\ + R(Q - q)(R - r),&\end{aligned}$$

$$\begin{aligned}0 = Pss + Rs(Q - q) + PRs(P - p) + R(P - p)(Qr - Rq)&\\ + Qs(R - r) - QQs(P - p) + R(R - r)^2,&\end{aligned}$$

which contain the required determinations.

24. If the two given equations are of an arbitrary order, as

$$\begin{aligned}z^m + Pz^{m-1} + Qz^{m-2} + Rz^{m-3} + \cdots &= 0,\\ z^n + pz^{n-1} + qz^{n-2} + rz^{n-3} + \cdots &= 0,\end{aligned}$$

and we wish to determine the relation between their coefficients so that these two equations would have two common roots, we will find by similar reasoning that it is necessary to satisfy this equation:

$$\begin{aligned}(z^m + Pz^{m-1} + Qz^{m-2} + \cdots)(z^{n-2} + az^{n-3} + bz^{n-4} + \cdots)&\\ = (z^n + pz^{n-1} + qz^{n-2} + \cdots)(z^{m-2} + Az^{m-3} + Bz^{m-4} + \cdots)&\end{aligned}$$

in such a way that the coefficients of each power of z become equal on either side.

25. Now, by setting these terms equal, we will obtain $m+n-2$ equalities. But the number of the unknown coefficients A, B, C, etc. is $m-2$ and the number of the others a, b, c, etc. is $n-2$, so we need only $m+n-4$ equalities in order to determine all these coefficients. Therefore, after determining all these unknown coefficients, we will still find two more equations among the coefficients P, Q, R, etc. and p, q, r, etc., which will contain the required conditions so that the two given equations will have two common roots. This determination will serve in the Theory of Curves to find the case where two curves intersect at two points in such a way that these two intersections correspond to the same abscissa, indicated by z.

26. After what I just said, it will not be difficult to find the conditions under which two equations of arbitrary degree will acquire three common roots. For, if the two given equations are

$$\begin{aligned} z^m + Pz^{m-1} + Qz^{m-2} + Rz^{m-3} + \cdots &= 0, \\ z^n + pz^{n-1} + qz^{n-2} + rz^{n-3} + \cdots &= 0, \end{aligned}$$

we have only to form this equation

$$\begin{aligned} &(z^m + Pz^{m-1} + Qz^{m-2} + \cdots)(z^{n-3} + az^{n-4} + bz^{n-5} + \cdots) \\ &\qquad = (z^n + pz^{n-1} + qz^{n-2} + \cdots)(z^{m-3} + Az^{m-4} + Bz^{m-5} + \cdots) \end{aligned}$$

and equate the coefficients of each power of z. This operation, after having determined the coefficients A, B, C, etc., a, b, c, etc., will lead to three equations among the given coefficients, which will contain the required conditions for these two equations to obtain three common roots.

27. From this it is clear enough how we would find the necessary determinations for two given equations to obtain four or more common roots; and these conditions will always be comprised of as many equations as there are roots that must be common to the given equations. Since the method which I just showed for this purpose is altogether similar to that which serves elimination, which is the case for two equations that must have one common root, I thought that it would merit some attention; and this all the more because the ordinary methods we use for elimination are not sufficient to solve the other problems I just related.