A solution to a curious problem which seems not subjected to any analysis

Leonhard Euler

§1. I found myself among company one day, where on the occasion of a game of chess someone suggested this problem: *to have a knight visit all the squares of a chessboard, never landing twice on the same square, by starting from a given square.* Chips were placed for this purpose on all 64 squares of the chessboard, except for the one where the knight had to start his route; and at each square where the knight landed while marching accordingly, the chip was removed, so that it was about successively picking up, in this way, all of the chips. It was required to avoid, on the one hand, the knight ever landing on an empty square, and on the other hand it was necessary to run the course in such a way that he visited all the squares by the end.

§2. Those who believed this to be an easy enough problem tried in vain several times but were unable to achieve the goal. After that, the one who had suggested the problem, having begun from the given square, knew very well how to run the route, and he happily picked up all the chips. However, the multitude of squares didn't allow committing to memory the route he had followed, and it was only after several tries that I finally encountered a route which solved the problem, yet of value only for a certain initial square. I do not remember more, whether we gave him the freedom to choose it himself; but he was very positively assured that he was in a position to execute, whichever square we wanted to start from.

§3. In order to clarify this problem, I will add a route here, where by starting from a corner of the chessboard, all the squares are visited:

Solution d'une question curieuse qui ne paroit soumise à aucune analyse, Mémoires de l'académie des sciences de Berlin **15** (1766), 310–337. Number 309 in the Eneström index.

42	59	44	9	40	21	46	7
61	10	41	58	45	8	39	20
12	43	60	55	22	57	6	47
53	62	11	30	25	28	19	38
32	13	54	27	56	23	48	5
63	52	31	24	29	26	37	18
14	33	2	51	16	35	4	49
1	64	15	34	3	50	17	36

Here I marked the squares in numerical order, following how they are successively visited. So, the knight is first set at square 1, jumps to square 2, from there to 3, and then 4, 5, 6, etc., until, coming finally to square 64, he will have visited all the squares. It is evident that this route works equally well if it is desired to start at one of the other corners.

§4. Returning by the same route, we will also be able to start from square 64, and from there, by successively visiting squares 63, 62, 61, etc., we will finally reach, after having visited all the squares, the corner square 1. But this route will not help at all when starting from some other square; and indeed we would have to search by trial and error for a new route starting from the given square. It is easily recognized that this kind of solution to the problem would be too hard, and would be ill-suited to the purpose at hand, which is about quickly finding the route which must be followed. Besides, such an investigation doesn't merit any attention unless it is based on some principles, or unless one could submit it to some kind of analysis which directs its operations.

§5. It is also only in this view that I venture to offer my investigations on this problem, to which I was led by a very particular idea given to me by Mr. Bertrand of Geneva. For although this problem is self-contained and altogether foreign to Geometry, it must be regarded as very remarkable once a way will have been found to apply Analysis to it. Indeed, I will show that it is susceptible to a very particular analysis, which merits all the more attention because this analysis demands reasoning which is not much used elsewhere. One easily recognizes the excellence of Analysis, but it is commonly believed to be limited to certain investigations related to Mathematics. Consequently, it will always be most important to make use of Analysis in matters which appear impenetrable to it, since it is certain that it encompasses the art of reasoning to the highest degree. One cannot extend the limits of Analysis, then, without being right to hope for very great advantages from so doing.

§6. To start, I note that the problem could be solved if a route were found where the last square, 64, is separated from the first, 1, by one knight jump, so that he could jump from the last to the first. For having found such a route which re-enters itself, we will be able to start from whichever square, and from there continue the course by following the numbers in order up to the square marked 64, where after jumping to square 1, he will finish the course by returning to the square from which he had left. Indeed, here is one such course which is re-entrant upon itself:

42	57	44	9	40	21	46	7
55	10	41	58	45	8	39	20
12	43	56	61	22	59	6	47
63	54	11	30	25	28	19	38
32	13	62	27	60	23	48	5
53	64	31	24	29	26	37	18
14	33	2	51	16	35	4	49
1	52	15	34	3	50	17	36

§7. So by memorizing well one route like this, we will be in a position to solve the problem starting from any square whatsoever. For example, suppose square 25 is the one the knight must leave from. Then we will have only to move him successively to squares 26, 27, 28, ..., up to square 64, then after skipping to square 1, his route will continue with the squares 2, 3, 4, ..., until he comes to square 24; and in this way he will have visited all the squares on the chessboard. I will indicate this route by the numbers which mark the squares, like this

$$25 \cdots 64 \cdot 1 \cdots 24,$$

and it is evident that we could succeed equally well when starting from any other square. Thus this arrangement

$$46 \cdots 64 \cdot 1 \cdots 45,$$

will serve when we must start from square 46.

§8. It is also evident that the same arrangement provides, for each starting square, two routes, since we could equally well visit the squares in reverse numerical order, down to square 1, and from there, after jumping to square 64, continue the course with squares 63, 62, 61, etc., until we reach the square we started from. Suppose 40 is the square we must leave from. Then we will have these two routes to follow:

$$40 \cdot 41 \cdots 64 \cdot 1 \cdot 2 \cdots 39$$

and

$$40 \cdot 39 \cdots 1 \cdot 64 \cdot 63 \cdots 41,$$

where the first finishes at square 39 and the second at 41.

§9. It is left for me to explain, then, a certain method which will unfailingly lead us to the goal described, and by means of which we will be in a position to discover as many routes which satisfy as we might want. For although the number of these routes is not infinite, there will always be so many that we could never lack for them. But here we must distinguish two kinds of routes. The first kind simply visits all the squares on the chessboard without the knight being able to jump from the last to the first. The second kind are the routes which are re-entrant upon themselves, where the the knight, after having visited all the squares, can jump from the last to the first. I have given an example of the first kind in §3, and an example of the second kind in §6. They can be thought of as found randomly by trail and error, but the method which I will explain will serve to find as many as desired, both of the first kind and of the second kind.

§10. Since it is much more difficult to find only by trial a route of the second kind, I will start by giving a method by means of which one will be able, after having found a route of the first kind, to discover from it not only one, but several of the second kind. For this purpose, I remark first that the last square can be changed, in several ways, while keeping the first square the same. Let us consider the route reported in §3, and note which squares the knight would be able to get to from the last square, marked 64. These squares are seen to be

$$63,\ 31,\ \text{and}\ 51.$$

The first is already included in the jump to 64 and is not of any use. But, since we can go from square 31 to square 64, let us make this jump after reaching square 31, after having passed through squares 1, 2, 3, 4, etc., and then let us continue the route with squares 64, 63, 62, etc., until we get to square 32, which will now be the final square. This new route will be represented as

$$1 \cdot 2 \cdots 31 \cdot 64 \cdot 63 \cdots 32.$$

§11. Similarly, the jump from 64 to 51 lets us know that we can go from square 51 to 64. From there, by continuing the route with squares 63, 62, etc., the last one will be square 52. This entire route will then be represented as

$$1 \cdot 2 \cdots 51 \cdot 64 \cdot 63 \cdots 52.$$

Now, since this last square, 52, allows a jump to the first, this is a route of the second kind, and is re-entrant upon itself. This is precisely the route described in §6.

Though if we hadn't yet reached a re-entrant route, we could again transform the one we just found in the preceding §

$$1 \cdots 31 \cdot 64 \cdots 32,$$

where the knight can jump from the last square, 32, to squares

$$43,\ 11,\ 31,\ 33.$$

We would have only to reverse the part of the route contained between one of these numbers and the final 32.

§12. The number 43 will supply this new route

$$1 \cdots 31 \cdot 64 \cdots 43 \cdot 32 \cdots 42,$$

where the corner square 42 is the last. The second number, 11, will give this route

$$1 \cdots 11 \cdot 32 \cdots 64 \cdot 31 \cdots 12,$$

where the square marked 12 is now the last. The third number, 31, yields the main route we have been using to derive these others, namely

$$1 \cdots 31 \cdot 32 \cdots 64,$$

and the fourth number, 33, doesn't change anything in the route we are treating.

In the preceding route which ended with 12, since the knight can jump from 12 to these squares

$$59, \ 41, \ 11, \ \text{and } 13,$$

the route will supply these transformed ones:

$$1 \cdots 11 \cdot 32 \cdots 59 \cdot 12 \cdots 31 \cdot 64 \cdots 60,$$

$$1 \cdots 11 \cdot 32 \cdots 41 \cdot 12 \cdots 31 \cdot 64 \cdots 42.$$

In the first, 60 leads to the squares

$$61, \ 59, \ 9, \ 45, \ 25, \ 27, \ 13, \ \text{and } 53,$$

and this will furnish several new routes, whose the final squares will be

$$10, \ 46, \ 26, \ 28, \ 14, \ \text{and } 54.$$

§13. So here is quite a rich source from which we can draw numerous new routes, after having found a single one; and the number of transformations becomes larger still when we reverse the order of the original route, as

$$64 \cdots 1.$$

Here the last square is attached to 52, and supplies this transform

$$64 \cdots 52 \cdot 1 \cdots 51,$$

and since 51 enables a jump to 64, this route is re-entrant upon itself. But this is only the reverse of the one above in §11. Now, 51 is connected to

$$64,\ 52,\ 54,\ 56,\ 26,\ \text{and}\ 50,$$

and so it supplies these transforms

$$64 \cdots 54 \cdot 51 \cdots 1 \cdot 52 \cdot 53,$$

$$64 \cdots 56 \cdot 51 \cdots 1 \cdot 52 \cdots 55,$$

$$64 \cdots 52 \cdot 1 \cdots 26 \cdot 51 \cdots 27,$$

and from these latter, if desired, numerous others can again be found, and one will not fail to discover among them those which are re-entrant among themselves.

§14. Now, having already found one route which is re-entrant upon itself, like the one in §6, it is not difficult to derive several others of the same nature. The squares need only be arranged so that both the first and last squares are away from the edges of the chessboard, so that they both allow eight jumps. In this way, if we arrange the numbers of the route in §6 as

$$31 \cdots 64 \cdot 1 \cdots 30,$$

the last square, 30, is joined to these

$$45,\ 59,\ 23,\ 29,\ 31,\ 13,\ 43,\ 41,$$

and yields these transforms

$$\begin{aligned} \text{I.}\quad & 31\cdots 45\cdot 30\cdots 1\cdot 64\cdots 46,\\ \text{II.}\quad & 31\cdots 59\cdot 30\cdots 1\cdot 64\cdots 60,\\ \text{III.}\quad & 31\cdots 64\cdot 1\cdots 23\cdot 30\cdots 24,\\ \text{IV.}\quad & 31\cdots 64\cdot 1\cdots 13\cdot 30\cdots 14,\\ \text{V.}\quad & 31\cdots 43\cdot 30\cdots 1\cdot 64\cdots 44,\\ \text{VI.}\quad & 31\cdots 41\cdot 30\cdots 1\cdot 64\cdots 42, \end{aligned}$$

where II and IV are re-entrant upon themselves. And, from these and the other ones, we will be able to find through subsequent transformations several more. For example III gives

$$31\cdots 64\cdot 1\cdots 13\cdot 24\cdots 30\cdot 23\cdots 14,$$
$$31\cdots 33\cdot 24\cdots 30\cdot 23\cdots 1\cdot 64\cdots 34,$$
$$31\cdots 64\cdot 1\cdots 15\cdot 24\cdots 30\cdot 23\cdots 16.$$

§15. But when we do not yet have any route of the first kind, let us see how to go about finding one without relying on chance. Starting from an arbitrary square, let us jump with the knight at will, as far as we can, and then place letters on the squares which remain empty, to serve as signs; as in this figure:

34	21	54	9	32	19	48	7
55	10	33	20	53	8	31	18
22	35	62	*a*	40	49	6	47
11	56	41	50	59	52	17	30
36	23	58	61	42	39	46	5
57	12	25	38	51	60	29	16
24	37	2	43	14	27	4	45
1	*b*	13	26	3	44	15	28

Here I was able to continue the route until the square marked 62, and then in the two remaining empty squares I placed the letters *a* and *b*.

§16. Now, after visiting 62 squares with the knight, I represent them in this manner

$$1 \cdots 62;$$

and regarding square 62 as the last, I look for a transform which ends on another square, and from which there is a passage to one of the squares a or b. Square 62 communicates with these

$$9,\ 53,\ 59,\ 61,\ 23,\ 11,\ 55,\ \text{and } 21,$$

which gives us these transforms

I. $1 \cdots 9 \cdot 62 \cdots 10$, which goes to a,

II. $1 \cdots 53 \cdot 62 \cdots 54$, which goes to a,

III. $1 \cdots 59 \cdot 62 \cdots 60$,

IV. $1 \cdots 23 \cdot 62 \cdots 24$,

V. $1 \cdots 11 \cdot 62 \cdots 12$,

VI. $1 \cdots 55 \cdot 62 \cdots 56$, which goes to a,

VII. $1 \cdots 21 \cdot 62 \cdots 22$.

So routes I, II, and VI already extend to a, and the only remaining empty square is b. In order to connect it to the others, we need only transform one of these three routes by the same method. We would operate similarly if there were several empty squares.

§17. Let us take the first transform,

$$1 \cdots 9 \cdot 62 \cdots 10 \cdot a,$$

whose last square, a, leads to

$$32,\ 8,\ 52,\ 42,\ 58,\ 56,\ 10,\ \text{and } 54.$$

Among these, square 58 will supply this transform

$$1 \cdots 9 \cdot 62 \cdots 58 \cdot a \cdot 10 \cdots 57.$$

The last square, 57, leads to square b, so now the knight will have visited all the squares, starting from 1 and ending at b,

$$1 \cdots 9 \cdot 62 \cdots 58 \cdot a \cdot 10 \cdots 57 \cdot b.$$

But this route is not re-entrant upon itself. To procure for it this advantage, let us search for new transforms. The last square, b, leads to these squares

$$57,\ 25,\ 43.$$

Among these, 25 gives this transform

$$1\cdots 9\cdot 62\cdots 58\cdot a\cdot 10\cdots 25\cdot b\cdot 57\cdots 26,$$

whose last square leads to

$$37,\ 25,\ 51,\ \text{and } 27.$$

None of these yields a route of the second kind. So let us take 43, which gives

$$1\cdots 9\cdot 62\cdots 58\cdot a\cdot 10\cdots 43\cdot b\cdot 57\cdots 44,$$

whose last square, 44, leads to

$$43,\ 51,\ 29,\ \text{and } 45,$$

and none of these immediately gives a route re-entrant upon itself.

§18. So it will be necessary to pass to new transforms; and in order that this may be be done more easily, it will be good to represent the given route of the first kind by the natural order of numbers:

40	27	60	9	38	25	54	7
61	16	39	26	59	8	37	24
28	41	10	15	46	55	6	53
17	62	47	56	13	58	23	36
42	29	14	11	48	45	52	5
63	18	31	44	57	12	35	22
30	43	2	49	20	33	4	51
1	64	19	32	3	50	21	34

The route is represented by

$$1\cdots 64,$$

and the last square, 64, leads to

$$63,\ 31,\ 49,$$

and we will have two transforms

$$\text{I.} \quad 1 \cdots 31 \cdot 64 \cdots 32,$$

$$\text{II.} \quad 1 \cdots 49 \cdot 64 \cdots 50,$$

because square 63 does not change anything in the given route.

§19. Since there are only two squares which border square 1, let us reverse these two transforms in order to get

$$\text{I.} \quad 32 \cdots 64 \cdot 31 \cdots 1,$$

$$\text{II.} \quad 50 \cdots 64 \cdot 49 \cdots 1.$$

And now the last square, 1, leads to

2 and 18,

and we derive these two new transforms

$$A. \quad 32 \cdots 64 \cdot 31 \cdots 18 \cdot 1 \cdots 17,$$

$$B. \quad 50 \cdots 64 \cdot 49 \cdots 18 \cdot 1 \cdots 17.$$

The last square, 17, leads to

16, 10, 14, 18,

and will obtain for us

$$C. \quad 32 \cdots 64 \cdot 31 \cdots 18 \cdot 1 \cdots 10 \cdot 17 \cdots 11,$$

$$D. \quad 50 \cdots 64 \cdot 49 \cdots 18 \cdot 1 \cdots 10 \cdot 17 \cdots 11,$$

$$E. \quad 32 \cdots 64 \cdot 31 \cdots 18 \cdot 1 \cdots 14 \cdot 17 \cdots 15,$$

$$F. \quad 50 \cdots 64 \cdot 49 \cdots 18 \cdot 1 \cdots 14 \cdot 17 \cdots 15.$$

The last square, 11, leads to

46, 58, 12, 20, 2, 18, 62, 10,

and gives

$$
\begin{aligned}
&G.\quad 32\cdots 46\cdot 11\cdots 17\cdot 10\cdots 1\cdot 18\cdots 31\cdot 64\cdots 47,\\
&H.\quad 50\cdots 64\cdot 49\cdots 46\cdot 11\cdots 17\cdot 10\cdots 1\cdot 18\cdots 45,\\
&I.\quad 32\cdots 58\cdot 11\cdots 17\cdot 10\cdots 1\cdot 18\cdots 31\cdot 64\cdots 59,\\
&K.\quad 50\cdots 58\cdot 11\cdots 17\cdot 10\cdots 1\cdot 18\cdots 49\cdot 64\cdots 59,\\
&L.\quad 32\cdots 64\cdot 31\cdots 20\cdot 11\cdots 17\cdot 10\cdots 1\cdot 18\cdot 19,\\
&M.\quad 50\cdots 64\cdot 49\cdots 20\cdot 11\cdots 17\cdot 10\cdots 1\cdot 18\cdot 19,\\
&N.\quad 32\cdots 64\cdot 31\cdots 18\cdot 1\cdot 2\cdot 11\cdots 17\cdot 10\cdots 3,\\
&O.\quad 50\cdots 64\cdot 49\cdots 18\cdot 1\cdot 2\cdot 11\cdots 17\cdot 10\cdots 3,\\
&P.\quad 32\cdots 62\cdot 11\cdots 17\cdot 10\cdots 1\cdot 18\cdots 31\cdot 64\cdot 63,\\
&Q.\quad 50\cdots 62\cdot 11\cdots 17\cdot 10\cdots 1\cdot 18\cdots 49\cdot 64\cdot 63.
\end{aligned}
$$

§20. Now E and F, whose last square, 15, leads to

38, 8, 58, 48, 14, 62, 16, 60,

will give us these transforms

$$
\begin{aligned}
&g.\quad 32\cdots 38\cdot 15\cdots 17\cdot 14\cdots 1\cdot 18\cdots 31\cdot 64\cdots 39,\\
&h.\quad 50\cdots 64\cdot 49\cdots 38\cdot 15\cdots 17\cdot 14\cdots 1\cdot 18\cdots 37,\\
&i.\quad 32\cdots 64\cdot 31\cdots 18\cdot 1\cdots 8\cdot 15\cdots 17\cdot 14\cdots 9,\\
&k.\quad 50\cdots 64\cdot 49\cdots 18\cdot 1\cdots 8\cdot 15\cdots 17\cdot 14\cdots 9,\\
&l.\quad 32\cdots 58\cdot 15\cdots 17\cdot 14\cdots 1\cdot 18\cdots 31\cdot 64\cdots 59,\\
&m.\quad 50\cdots 58\cdot 15\cdots 17\cdot 14\cdots 1\cdot 18\cdots 49\cdot 64\cdots 59,\\
&n.\quad 32\cdots 48\cdot 15\cdots 17\cdot 14\cdots 1\cdot 18\cdots 31\cdot 64\cdots 49,\\
&o.\quad 50\cdots 64\cdot 49\cdot 48\cdot 15\cdots 17\cdot 14\cdots 1\cdot 18\cdots 47,\\
&p.\quad 32\cdots 62\cdot 15\cdots 17\cdot 14\cdots 1\cdot 18\cdots 31\cdot 64\cdot 63,\\
&q.\quad 50\cdots 62\cdot 15\cdots 17\cdot 14\cdots 1\cdot 18\cdots 49\cdot 64\cdot 63,\\
&r.\quad 32\cdots 60\cdot 15\cdots 17\cdot 14\cdots 1\cdot 18\cdots 31\cdot 64\cdots 61,\\
&s.\quad 50\cdots 60\cdot 15\cdots 17\cdot 14\cdots 1\cdot 18\cdots 49\cdot 64\cdots 61.
\end{aligned}
$$

But among all these transforms, still not a single one is found which is re-entrant upon itself. But their more distant transforms will suffice.

§21. Let us take the route indicated by the letter G, whose last square, 47, communicates with these

$$26,\ 46,\ 48,\ 44,\ 18,\ 42,\ 28,\ 16,$$

and the last squares obtained from these transformations will be

$$27,\ 11,\ 47,\ 45,\ 19,\ 43,\ 29,\ 17,$$

and among these, 43 communicates with the first, 32, and consequently gives this re-entrant route

$$32 \cdots 42 \cdot 47 \cdots 64 \cdot 31 \cdots 18 \cdot 1 \cdots 10 \cdot 17 \cdots 11 \cdot 46 \cdots 43,$$

which can then be represented as

$$1 \cdots 10 \cdot 17 \cdots 11 \cdot 46 \cdots 43 \cdot 32 \cdots 42 \cdot 47 \cdots 64 \cdot 31 \cdots 18.$$

Then marking the squares by the natural order of numbers, we will have this re-entrant route:

30	55	46	9	28	57	40	7
47	12	29	56	45	8	27	58
54	31	10	13	18	41	6	39
11	48	33	42	15	44	59	26
32	53	14	17	34	19	38	5
49	64	51	20	43	16	25	60
52	21	2	35	62	23	4	37
1	50	63	22	3	36	61	24

§22. The route indicated by the letter H has 45 for its last square, and the communicating squares are

$$6,\ 36,\ 22,\ 4,\ 20,\ 44,\ 56,\ 46,$$

and the last squares will be

$$5,\ 37,\ 23,\ 3,\ 21,\ 45,\ 57,\ 11,$$

and among these, 57 communicates with the first square, 50, resulting in this re-entrant route

$$50 \cdots 56 \cdot 45 \cdots 18 \cdot 1 \cdots 10 \cdot 17 \cdots 11 \cdot 46 \cdots 49 \cdot 64 \cdots 57,$$

which can again be represented as

$$1 \cdots 10 \cdot 17 \cdots 11 \cdot 46 \cdots 49 \cdot 64 \cdots 57 \cdot 50 \cdots 56 \cdot 45 \cdots 18,$$

42	55	26	9	44	57	34	7
25	12	43	56	27	8	45	58
54	41	10	13	18	35	6	33
11	24	19	36	15	28	59	46
40	53	14	17	20	37	32	5
23	64	51	38	29	16	47	60
52	39	2	21	62	49	4	31
1	22	63	50	3	30	61	48

which does not differ greatly from the preceding.

§23. The routes indicated by I and K end with square 59, which communicates with these squares

$$54,\ 6,\ 58,\ 56,\ 10,\ 60.$$

The final squares for I will be

$$55,\ 5,\ 11,\ 57,\ 9,\ 59,$$

and the final squares for K

$$55,\ 5,\ 11,\ 57,\ 9,\ 59,$$

from which we derive two more re-entrant routes, since the 57 communicates with both 32 and 50:

$$32 \cdots 56 \cdot 59 \cdots 64 \cdot 31 \cdots 18 \cdot 1 \cdots 10 \cdot 17 \cdots 11 \cdot 58 \cdot 57,$$

$$50 \cdots 56 \cdot 59 \cdots 64 \cdot 49 \cdots 18 \cdot 1 \cdots 10 \cdot 17 \cdots 11 \cdot 58 \cdot 57.$$

These can be represented as

$$1 \cdots 10 \cdot 17 \cdots 11 \cdot 58 \cdot 57 \cdot 32 \cdots 56 \cdot 59 \cdots 64 \cdot 31 \cdots 18,$$

$$1 \cdots 10 \cdot 17 \cdots 11 \cdot 58 \cdot 57 \cdot 50 \cdots 56 \cdot 59 \cdots 64 \cdot 49 \cdots 18.$$

Similarly, routes L and M end with square 19, which communicates with these squares

30, 18, 44, 20.

The last squares for L are

29, 19, 45, 11,

and the last for M are

29, 19, 43, 11,

and there are not any re-entrant routes.

The routes N and O end with square 3, and this communicates with

2, 44, 12, 4.

Then the last becomes, for N:

11, 45, 13, 3;

and for O:

11, 43, 13, 3,

and there are none there either.

§24. If it would be worth the trouble, we could, by following these transformations, find several other re-entrant routes, and we would not fail to discover ways to abridge the operations, completing two or more at a time, in order to reach the goal more quickly. However it is not my plan to determine all the possible routes which are re-entrant upon themselves, which would be a labor as difficult as it were pointless; and I content myself having given a sure method for finding as many routes as desired, a method whose application is not difficult in each case. But we could add to the main problem more conditions which make it more curious, like if it is required that the numbers found on opposing squares have the same difference, which must be 32, as that is half the total number of squares. Each square has another which is opposed to it, so that a straight line drawn through the centers of the two squares divides the board into two equal parts. It is required, then, that the numbers 33, 34, 35, 36 $\cdots$ 64 are found opposite the numbers 1, 2, 3, 4 $\cdots$ 32.

§25. To find such diagonal routes, we have only to start by writing the numbers 1, 2, 3, 4, etc. corresponding to the march of the knight, and as we write these numbers, we put the numbers 33, 34, 35, 36, etc. in the opposite squares, and continue this arrangement as far as possible, as can be seen in the attached figure:

10	29	48	35	8	31	46	33
49	36	9	30	47	34	7	58
28	11	*A*	*C*	*f*	45	32	19
37	50	*B*	*D*	*e*	6	59	44
12	27	38	*E*	*d*	*b*	18	5
51	64	13	*F*	*c*	*a*	43	60
26	39	2	15	62	41	4	17
1	14	63	40	3	16	61	42

Here, I was able to continue the sequence of numbers 1, 2, 3, until 19, and the sequence numbers 33, 34, 35 until 51. But by going in reverse, I went from 1 to 64, 63, until 58; and from 33 I was able to move back until 26. Twelve squares remained empty, and I filled them with the letters A, a, B, b, C, c, D, d, E, e, F, f, arranged by opposing squares.

§26. We have, then, two separate sequences of squares which follow according to the march of the knight,

$$58 \cdots 64 \cdot 1 \cdots 19,$$

$$26 \cdots\cdots\cdots 51.$$

The square 19 communicates with 6, and so we will have these transforms, which can be further continued:

$$58 \cdots 64 \cdot 1 \cdots 6 \cdot 19 \cdots\cdot 7 \cdot f \cdot B \cdot d \cdot C,$$

$$26 \cdots\cdots\cdots 38 \cdot 51 \cdots 39 \cdot F \cdot b \cdot D \cdot c\,.$$

Now, the square C communicates with squares 8, 6, and d from the first sequence, so it does not provide any new transformations. But let us cut back the last two, and since it suffices to transform one sequence only, because the other is determined by it, let us take the first

$$58 \cdots 64 \cdot 1 \cdots 6 \cdot 19 \cdots 7 \cdot f \cdot B,$$

where B communicates with 12, and gives this transformation to continue

$$58 \cdots 64 \cdot 1 \cdots 6 \cdot 19 \cdots 12 \cdot B \cdot f \cdot 7 \cdots 11 \cdot D \cdot c\,.$$

The square c communicates with 16, and we will have

$$58 \cdots 64 \cdot 1 \cdots 6 \cdot 19 \cdots 16 \cdot c \cdot D \cdot 11 \cdots 7 \cdot f \cdot B \cdot 12 \cdots 15 \cdot a \cdot E,$$

and the other sequence will be

$$26 \cdots 38 \cdot 51 \cdots 48 \cdot C \cdot d \cdot 43 \cdots 39 \cdot F \cdot b \cdot 44 \cdots 47 \cdot A \cdot e,$$

where all the squares are included.

§27. Now it is necessary to link these two sequences together, so that the end of one touches the beginning of the other. For this purpose, let us transform the first sequence, whose end E communicates with square 62. The end then becomes 63, and will be coherent with the beginning of the other sequence, 26. So this transformation gives

$$58 \cdots 62 \cdot E \cdot a \cdot 15 \cdots 12 \cdot B \cdot f \cdot 7 \cdots 11 \cdot D \cdot c \cdot 16 \cdots 19 \cdot 6 \cdots 1 \cdot 64 \cdot 63,$$
$$26 \cdots 30 \cdot e \cdot A \cdot 47 \cdots 44 \cdot b \cdot F \cdot 39 \cdots 43 \cdot d \cdot C \cdot 48 \cdots 51 \cdot 38 \cdots\cdots 31,$$

and we have at the same time a route which is re-entrant upon itself, and also endowed with the prescribed condition:

14	59	42	35	16	31	54	33
41	36	15	58	55	34	17	30
60	13	56	43	18	53	32	7
37	40	19	12	57	6	29	52
20	61	38	25	44	51	8	5
39	64	21	50	11	24	45	28
62	49	2	23	26	47	4	9
1	22	63	48	3	10	27	46

§28. Having found a single route of this nature, it is easy to transform it in several different ways while preserving this same property. For in whatever manner we divide the re-entrant sequence of numbers $1 \cdots 64$ into two halves, one always contains the opposing squares of the other, as can be seen in these bisections:

$1 \cdots 32$	$2 \cdots\cdots 33$	$3 \cdots\cdots\cdots 34$	$4 \cdots\cdots\cdots\cdots 35$
$33 \cdots 64$	$34 \cdots 64 \cdot 1$	$35 \cdots 64 \cdot 1 \cdot 2$	$36 \cdots 64 \cdot 1 \cdot 2 \cdot 3$

where the two halves are always coherent. Now one has only to take such a bisection at will and transform the two halves similarly, until they become coherent again. In this way, let us take the half $3 \cdots 34$, whose end, 34, communicates with 7, and this gives the transform

$$3 \cdots 7 \cdot 34 \cdots 8,$$

and by reversing

$$8 \cdots 34 \cdot 7 \cdots 3,$$

whose end, 3, communicates with 24, and gives

$$8 \cdots 24 \cdot 3 \cdots 7 \cdot 34 \cdots 25,$$

and the other half will be

$$40 \cdots 56 \cdot 35 \cdots 39 \cdot 2 \cdot 1 \cdot 64 \cdots 57.$$

These are coherent at their ends 25, 40, and 8, 57. We will then be able to represent this new route as

$$1 \cdot 2 \cdot 39 \cdots 35 \cdot 56 \cdots 40 \cdot 25 \cdots 32,$$

$$33 \cdot 34 \cdot 7 \cdots \cdot 3 \cdot 24 \cdots 8 \cdot 57 \cdots 64.$$

§29. In this same half, $3 \cdots 34$, the first end, 3, communicates with 24, and gives this transform

$$23 \cdots 3 \cdot 24 \cdots 34,$$

and 34, which communicates with 7, gives

$$23 \cdots 7 \cdot 34 \cdots 24 \cdot 3 \cdots 6,$$

and the other half will be

$$55 \cdots 39 \cdot 2 \cdot 1 \cdot 64 \cdots 56 \cdot 35 \cdots 38,$$

which is coherent. Consequently, we will have a route represented by these two halves:

$$1 \cdot 2 \cdot 39 \cdots 55 \cdot 6 \cdots \cdot 3 \cdot 24 \cdots 32,$$

$$33 \cdot 34 \cdot 7 \cdots 23 \cdot 38 \cdots 35 \cdot 56 \cdots 64.$$

The half $4 \cdots 35$, because of the communication between the end 35 and the square 18, gives

$$4 \cdots 18 \cdot 35 \cdots 19,$$

which is already coherent with

$$36 \cdots 50 \cdot 3 \cdot 2 \cdot 1 \cdot 64 \cdots 51,$$

from which we derive this route:

$$1 \cdot 2 \cdot 3 \cdot 50 \cdots 36 \cdot 19 \cdots 32,$$

$$33 \cdot 34 \cdot 35 \cdot 18 \cdots 4 \cdot 51 \cdots 64,$$

and from other transformations the same half gives

$$1 \cdot 2 \cdot 3 \cdot 50 \cdots 43 \cdot 36 \cdot 19 \cdots 23 \cdot 10 \cdots 5 \cdot 24 \cdots 32,$$

$$33 \cdot 34 \cdot 35 \cdot 18 \cdots 11 \cdot 4 \cdot 51 \cdots 55 \cdot 42 \cdots 37 \cdot 56 \cdots 64.$$

§30. Here then are four other routes which have the same property as the one in §27.

50	59	22	7	48	31	10	33
23	6	49	58	9	34	47	30
60	51	8	21	46	11	32	35
5	24	45	52	57	36	29	12
44	61	4	25	20	13	56	37
3	64	43	14	53	40	19	28
62	15	2	41	26	17	38	55
1	42	63	16	39	54	27	18

42	59	6	55	44	31	18	33
5	54	43	58	19	34	45	30
60	41	56	7	46	17	32	35
53	4	47	40	57	20	29	16
48	61	52	25	8	15	36	21
3	64	49	14	39	24	9	28
62	13	2	51	26	11	22	37
1	50	63	12	23	38	27	10

40	59	12	35	38	31	54	33
13	18	39	58	55	34	37	30
60	41	56	11	36	53	32	47
17	14	19	42	57	48	29	52
20	61	16	25	10	51	46	49
15	64	21	4	43	24	9	28
62	5	2	23	26	7	50	45
1	22	63	6	3	44	27	8

40	59	50	35	38	31	48	33
51	12	39	58	49	34	37	30
60	41	56	11	36	47	32	21
55	52	13	42	57	22	29	46
14	61	54	25	10	45	20	23
53	64	15	4	43	24	9	28
62	5	2	17	26	7	44	19
1	16	63	6	3	18	27	8

§31. To this condition about opposing squares, we could add this one, that the first half of numbers, $1 \cdots 32$, fill only half the board,

α								33	β
	1	a	b	28	7	14	19	16	
	24	27	8	c	20	17	6	13	
	9	2	25	22	11	4	15	18	
	26	23	10	3	d	21	12	5	

by dividing the board with a line parallel to a side, so that the numbers $1 \cdots 32$ are all found below the line $\alpha\beta$, and the others, $33 \cdots 64$, are found above it. It is then necessary that unity be found near the line $\alpha\beta$, so that it may communicate with the number 64, which is found above the line.

§32. Let us start by putting unity at such a square, and by virtue of opposition, square 33 will also be determined. It will be necessary to make sure that it communicates with square 32 below the line $\alpha\beta$. While attempting such an arrangement, I only reached number 28, and I wrote in the empty squares the letters a, b, c, d, for the disposition of which I make the following transformations. In the sequence $1 \cdots 28$, square 28 touches

$$27,\ 25,\ 11,\ 17,$$

and we get these transformations

$$\text{I.}\quad 1 \cdots 25 \cdot 28 \cdots 26,$$

$$\text{II.}\quad 1 \cdots 11 \cdot 28 \cdots 12,$$

$$\text{III.}\quad 1 \cdots 17 \cdot 28 \cdots 18,$$

none of which extend to any of the empty squares. But after several transformations, the following is reached, which includes all the squares

$$1 \cdots 8 \cdot 23 \cdots \cdot 21 \cdot 18 \cdots \cdot 20 \cdot b \cdot 24 \cdots 28 \cdot 17 \cdots 9 \cdot a \cdot c \cdot d,$$

which is finally transformed into the following

$$1 \cdots 8 \cdot 23 \cdots 21 \cdot 18 \cdots 20 \cdot b \cdot 24 \cdots 28 \cdot 17 \cdots 15 \cdot d \cdot c \cdot a \cdot 9 \cdots 14,$$

whose final square, 14, communicates with the beginning, 33, of the other half above the line $\alpha\beta$; and the final square of the latter, 64, will itself communicate with square 1.

§33. Here is this route, then, represented in its entirety

	37	62	43	56	35	60	41	50	
	44	55	36	61	42	49	34	59	
	63	38	53	46	57	40	51	48	
	54	45	64	39	52	47	58	33	
α									β
	1	26	15	20	7	32	13	22	
	16	19	8	25	14	21	6	31	
	27	2	17	10	29	4	23	12	
	18	9	28	3	24	11	30	5	

and it is not only easy to find by the same method several others, but one can also transform this one in several ways, among them

$$7 \cdots 1 \cdot 8 \cdots 32,$$

$$7 \cdots 1 \cdot 8 \cdots 25 \cdot 32 \cdots 26,$$

$$15 \cdots 10 \cdot 7 \cdots 1 \cdot 8 \cdot 9 \cdot 16 \cdots 21 \cdot 24 \cdots 32 \cdot 23 \cdot 22,$$

which one could again reverse, just like the original, representing it as

$$32 \cdots 1.$$

§34. Here then are four more routes of this kind.

	35	62	43	56	37	60	41	50	
	44	55	36	61	42	49	38	59	
	63	34	53	46	57	40	51	48	
	54	45	64	33	52	47	58	39	
α									β
	7	26	15	20	1	32	13	22	
	16	19	8	25	14	21	2	31	
	27	6	17	10	29	4	23	12	
	18	9	28	5	24	11	30	3	

	35	60	43	56	37	62	41	50	
	44	55	36	61	42	49	38	63	
	59	34	53	46	57	40	51	48	
	54	45	58	33	52	47	64	39	
α									β
	7	32	15	20	1	26	13	22	
	16	19	8	25	14	21	2	27	
	31	6	17	10	29	4	23	12	
	18	9	30	5	24	11	28	3	

41	60	37	54	43	58	47	50
36	63	42	59	38	49	44	57
61	40	53	34	55	46	51	48
64	35	62	39	52	33	56	45
13	24	1	20	7	30	3	32
16	19	14	23	2	21	8	29
25	12	17	6	27	10	31	4
18	15	26	11	22	5	28	9

62	37	56	41	60	35	54	47
57	42	61	36	55	48	51	34
38	63	44	59	40	53	46	49
43	58	39	64	45	50	33	52
20	1	18	13	32	7	26	11
17	14	21	8	27	12	31	6
2	19	16	23	4	29	10	25
15	22	3	28	9	24	5	30

§35. Up until now I've considered the problem as posed: for an ordinary chessboard divided into 64 squares. But this number is too large to allow one to conceive of all the varieties which may occur, so it would be good to also consider simpler figures that contain fewer squares which the knight must visit. It is immediately clear that neither a square board of 4 nor a square board of 9 is suitable; but it may be seen that a square board of 16 will not work either. For however one may go about it, an empty corner square will always remain; and it will soon be discovered that all the transformations which one may do are not capable of filling it. It is clear that one would have to start and finish in a corner; and consequently two of the four middle squares will be initially filled, and the two others would need to be guarded until the end, which is not possible.

1	8	13	10
14	11	4	7
5	2	9	12
	15	6	3

§36. So the first square board which the knight can cover is the one with 25 squares, which can be filled by means of the same rules, in case one does not succeed on the first try. Now, the march of the knight always produces this property, that the even and odd numbers follow each other alternately on the board, as can be seen in all the figures reported until now. From this it is evident that the last square, containing 25, can never communicate with the first square, 1; and consequently it is impossible to find a route which is re-entrant upon itself on the square board of 25, nor in any other figure which contains an odd number of squares. It is also understood

7	12	17	22	5
18	23	6	11	16
13	8	25	4	21
24	19	2	15	10
1	14	9	20	3

from this that one could never start on a square which contains an even number; for in whatever manner this route is transformed, the even numbers will always fall on the same squares, and the corner squares will contain odd numbers. In this square board of 25, it is also clear that it is absolutely necessary to either start or finish on a corner square.

§37. But let us also see the transformations that can be derived from this route $1 \cdots 25$ found on the board with 25 squares. The last square communicates with the squares

$$20,\ 10,\ 16,\ 22,\ 12,\ 18,\ 24,\ 14$$

and supplies these transforms

$$\begin{aligned} &\text{I.} && 1 \cdots 20 \cdot 25 \cdots 21, \\ &\text{II.} && 1 \cdots 10 \cdot 25 \cdots 11, \\ &\text{III.} && 1 \cdots 16 \cdot 25 \cdots 17, \\ &\text{IV.} && 1 \cdots 22 \cdot 25 \cdots 23, \\ &\text{V.} && 1 \cdots 12 \cdot 25 \cdots 13, \\ &\text{VI.} && 1 \cdots 18 \cdot 25 \cdots 19, \\ &\text{VII.} && 1 \cdots 25, \\ &\text{VIII.} && 1 \cdots 14 \cdot 25 \cdots 15. \end{aligned}$$

So, starting with the corner square, we can finish at one of the squares

$$21,\ 11,\ 17,\ 23,\ 13,\ 19,\ 25,\ 15.$$

But the first gives these additional transforms

$$\begin{aligned} &a. && 1 \cdots 6 \cdot 21 \cdots 25 \cdot 20 \cdots 7, \\ &b. && 1 \cdot 2 \cdot 21 \cdots 25 \cdot 20 \cdots 3, \end{aligned}$$

and the others give these transforms

$$\begin{aligned}
c.&\quad 1\cdot 2\cdot 11\cdots 25\cdot 10\cdots 3,\\
d.&\quad 1\cdots 8\cdot 11\cdots 25\cdot 10\cdots 9,\\
e.&\quad 1\cdots 4\cdot 17\cdots 25\cdot 16\cdots 5,\\
f.&\quad 1\cdots 8\cdot 17\cdots 25\cdot 16\cdots 9,\\
g.&\quad 1\cdots 4\cdot 23\cdots 25\cdot 22\cdots 5,\\
h.&\quad 1\cdot 2\cdot 23\cdots 25\cdot 22\cdots 3,\\
i.&\quad 1\cdots 6\cdot 13\cdots 25\cdot 12\cdots 7,\\
k.&\quad 1\cdot 2\cdot 13\cdots 25\cdot 12\cdots 3,\\
l.&\quad 1\cdots 6\cdot 19\cdots 25\cdot 18\cdots 7,\\
m.&\quad 1\cdots 4\cdot 19\cdots 25\cdot 18\cdots 5,\\
n.&\quad 1\cdots 6\cdot 15\cdots 25\cdot 14\cdots 7,\\
o.&\quad 1\cdots 8\cdot 15\cdots 25\cdot 14\cdots 9,
\end{aligned}$$

where the last squares are 3, 5, 7, and 9.

§38. Since the corner squares 3, 5, 7 communicate only with two others, they do not yield by our method any new transforms. Let us consider then those which finish on 9, and we will derive these transforms

$$\begin{aligned}
p.&\quad 1\cdots 4\cdot 9\cdot 10\cdot 25\cdots 11\cdot 8\cdots 5,\\
q.&\quad 1\cdots 8\cdot 11\cdots 24\cdot 9\cdot 10\cdot 25,\\
r.&\quad 1\cdots 4\cdot 9\cdots 16\cdot 25\cdots 17\cdot 8\cdots 5,\\
s.&\quad 1\cdots 8\cdot 17\cdots 24\cdot 9\cdots 16\cdot 25,\\
t.&\quad 1\cdots 4\cdot 9\cdots 14\cdot 25\cdots 15\cdot 8\cdots 5,\\
u.&\quad 1\cdots 8\cdot 15\cdots 24\cdot 9\cdots 14\cdot 25.
\end{aligned}$$

Now these new routes which finish on 25 lead us to other transforms, and we will reach several other routes which finish on odd-numbered squares. From this we see that by starting at the corner square 1, we can finish at whatever odd-numbered square we wish, and this in several different ways. In turn, each route can be reversed, and so the total number of possible routes will become very large.

§39. Here we can add this additional condition: that the numbers found on two opposing squares always add up to the same number, namely 26. It is then necessary that the first and last squares be found at opposite corners. To find such a route, one has only to begin filling the board and putting at each number's opposite its complement to 26, and to continue however far one can. But since it is known that the middle square must contain 13, one can hardly fail, and indeed, while maintaining the same property, several different forms can be derived. Here are some:

23	18	5	10	25
6	11	24	19	14
17	22	13	4	9
12	7	2	15	20
1	16	21	8	3

I. $1 \cdots 4 \cdot 11 \cdots 5 \cdot 14 \cdot 13 \cdot 12 \cdot 21 \cdots 15 \cdot 22 \cdots 25$,

II. $1 \cdots 4 \cdot 7 \cdots 5 \cdot 14 \cdots 18 \cdot 13 \cdot 8 \cdots 12 \cdot 21 \cdots 19 \cdot 22 \cdots 25$,

III. $1 \cdots 4 \cdot 21 \cdots 14 \cdot 13 \cdot 12 \cdots 5 \cdot 22 \cdots 25$,

IV. $1 \cdots 5 \cdot 14 \cdots 20 \cdot 13 \cdot 6 \cdots 12 \cdot 21 \cdots 25$,

V. $1 \cdots 4 \cdot 11 \cdot 12 \cdot 21 \cdots 16 \cdot 13 \cdot 10 \cdots\cdots 5 \cdot 14 \cdot 15 \cdot 22 \cdots 25$,

VI. $1 \cdots 4 \cdot 7 \cdots 12 \cdot 21 \cdot 20 \cdot 13 \cdot 6 \cdot 5 \cdot 14 \cdots 19 \cdot 22 \cdots 25$,

VII. $1 \cdots 4 \cdot 21 \cdot 12 \cdots 6 \cdot 13 \cdot 20 \cdots 14 \cdot 5 \cdot 22 \cdots 25$.

§40. In each of these variations, both the first four numbers, $1 \ldots 4$, and the last four, $22 \ldots 25$, along with the middle, 13, remain invariable, so that all the variations extend only to the others. From this it also seems that the given route, along with its seven variations, entirely exhausts this type. Here then are all eight of these routes represented together at once.

23	18	5	10	25
6	11	24	19	14
17	22	13	4	9
12	7	2	15	20
1	16	21	8	3

23	18	11	6	25
10	5	24	17	12
19	22	13	4	7
14	9	2	21	16
1	20	15	8	3

23	12	7	16	25
6	17	24	21	8
11	22	13	4	15
18	5	2	9	20
1	10	19	14	3

23	8	21	16	25
20	15	24	7	12
9	22	13	4	17
14	19	2	11	6
1	10	5	18	3

23	10	19	14	25
18	5	24	9	20
11	22	13	4	15
6	17	2	21	8
1	12	7	16	3

23	20	15	8	25
14	9	24	21	16
19	22	13	4	7
10	5	2	17	12
1	18	11	6	3

23	16	21	8	25
12	7	24	15	20
17	22	13	4	9
6	11	2	19	14
1	18	5	10	3

23	10	5	18	25
14	19	24	11	6
9	22	13	4	17
20	15	2	7	12
1	8	21	16	3

§41. The routes found above for a square board of 25 can be arranged to fill a square board of 100, in such a way that the route becomes re-entrant upon itself. Here is such a square board of 100,

30	41	46	37	32	53	60	67	72	55
47	36	31	40	45	68	73	54	61	66
42	29	38	33	50	59	52	63	56	71
35	48	27	44	39	74	69	58	65	62
28	43	34	49	26	51	64	75	70	57
7	20	25	14	1	76	99	84	93	78
12	15	8	19	24	89	94	77	98	85
21	6	13	2	9	100	83	88	79	92
16	11	4	23	18	95	90	81	86	97
5	22	17	10	3	82	87	96	91	80

where the numbers are arranged in four quarters, each of which contains the same route.

§42. Before finishing, I will again add some more figures. Among the rectangles, the simplest which the knight can cover has 12 squares, the height containing 3 and the width containing 4. Here are some of those routes:

10	7	2	5
1	4	9	12
8	11	6	3

3	6	11	8
12	9	2	5
1	4	7	10

3	6	9	12
8	11	2	5
1	4	7	10

12	9	6	3
1	4	11	8
10	7	2	5

But it is easily seen that the re-entrant routes can have no place here.

If the height contains 3 squares and the width 5 or 6, it is impossible to visit them all. But by making the width have 7 or more squares, one will be able to succeed, however without re-entrancy:

3	8	5	18	15	10	13
6	19	2	9	12	21	16
1	4	7	20	17	14	11

15	18	21	2	5	8	11
20	1	16	13	10	3	6
17	14	19	4	7	12	9

If we make the height 4 squares, and the width 5 or more, then we will have these routes:

14	7	20	3	16
19	2	15	8	11
6	13	10	17	4
1	18	5	12	9

16	7	22	3	18	11
23	2	17	12	21	4
8	15	6	19	10	13
1	24	9	14	5	20

20	7	26	13	18	5	24
27	14	19	6	25	12	17
8	21	2	15	10	23	4
1	28	9	22	3	16	11

§43. Up to now, the re-entrant routes could have no place; but by making the height 5 squares and the width 6, this condition can also be satisfied, the same way as with all the other rectangles that have an even number of squares, provided there are at least 5 squares on a side. Here are some examples:

3	20	13	24	5	18
12	29	4	19	14	25
21	2	23	8	17	6
28	11	30	15	26	9
1	22	27	10	7	16

30	21	6	15	28	19
7	16	29	20	5	14
22	31	8	35	18	27
9	36	17	26	13	4
32	23	2	11	34	25
1	10	33	24	3	12

where this latter figure is a square board of 36, and the route is not only re-entrant upon itself, but all the opposing squares differ by 18.

§44. But without being limited to rectangular shapes, one can form, as desired, numerous other shapes where the knight can visit all the squares. I will add some of the simplest ones, which even allow routes which are re-entrant upon themselves.

	10	7	
12	5	2	9
3	8	11	6
	1	4	

		14	19		
		7	12		
6	13	20	15	18	11
1	8	5	10	3	16
		2	17		
		9	4		

	1	14	7	22	
15	8	21	32	13	24
2	31	26	23	6	19
9	16	29	20	25	12
30	3	10	27	18	5
	28	17	4	11	

	1	20	7	26	
21	8	27	32	19	14
2	29	12	15	6	25
9	22	31	28	13	18
30	3	16	11	24	5
	10	23	4	17	